WEB CHAPTER

A

Optimization Techniques

WEB CHAPTER PREVIEW Normative economic decision analysis involves determining the action that best achieves a desired goal or objective. This means finding the action that optimizes (that is, maximizes or minimizes) the value of an objective function. For example, in a price-output decision-making problem, we may be interested in determining the output level that maximizes profits. In a production problem, the goal may be to find the combination of inputs (resources) that minimizes the cost of producing a desired level of output. In a capital budgeting problem, the objective may be to select those projects that maximize the net present value of the investments chosen. There are many techniques for solving optimization problems such as these. This chapter (and appendix) focuses on the use of differential calculus to solve certain types of optimization problems. In Web Chapter B, linear-programming techniques, used in solving constrained optimization problems, are examined. Optimization techniques are a powerful set of tools that are important in efficiently managing an enterprise's resources and thereby maximizing shareholder wealth.

MANAGERIAL CHALLENGE

A Skeleton in the Stealth Bomber's Closet[1]

In 1990 the U.S. Air Force publicly unveiled its newest long-range strategic bomber—the B-2 or "Stealth" bomber. This plane is characterized by a unique flying wing design engineered to evade detection by enemy radar. The plane has been controversial because of its high cost. However, a lesser-known controversy relates to its fundamental design.

The plane's flying wing design originated from a secret study of promising military technologies that was undertaken at the end of World War II. The group of prominent scientists who undertook the study concluded that a plane can achieve maximum range if it has a design in which virtually all the volume of the plane is contained in the wing. A complex mathematical appendix was attached to the study that purported to show that range could be *maximized* with the flying wing design.

However, a closer examination of the technical appendix by Joseph Foa, now an emeritus professor of engineering at George Washington University, discovered that a fundamental error had been made in the initial report. It turned out that the original researchers had taken the first derivative of a complex equation for the range of a plane and found that it had two solutions. The original researchers mistakenly concluded that the all-wing design was the one that maximized range, when, in fact, it *minimized* range.

In this chapter we introduce some of the same optimization techniques applied to an analysis of the Stealth bomber project. We develop tools designed to maximize profits or minimize costs. Fortunately, the mathematical functions we deal with in this chapter and throughout the book are much simpler than those that confronted the original "flying wing" engineers. We introduce techniques that can be used to check whether a function, such as profits or costs, is being minimized or maximized at a particular level of output.

[1]This Managerial Challenge is based primarily on W. Biddle, "Skeleton Alleged in the Stealth Bomber's Closet," *Science*, 12 May 1989, pp. 650–651.

TYPES OF OPTIMIZATION TECHNIQUES

In Chapter 1 we defined the general form of a problem that managerial economics attempts to analyze. The basic form of the problem is to identify the alternative means of achieving a given objective and then to select the alternative that accomplishes the objective in the most efficient manner, subject to constraints on the means. In programming terminology, the problem is optimizing the value of some objective function, subject to any resource and/or other constraints such as legal, input, environmental, and behavioral restrictions.

Mathematically, we can represent the problem as

$$\text{Optimize } y = f(x_1, x_2, \ldots, x_n) \qquad \text{[A.1]}$$

$$\text{subject to } g_j(x_1, x_2, \ldots, x_n) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} b_j \qquad j = 1, 2, \ldots, m \qquad \text{[A.2]}$$

where Equation A.1 is the objective function and Equation A.2 constitutes the set of constraints imposed on the solution. The x_i variables, $x_1, x_2, \ldots, x_n$, represent the set of decision variables, and $y = f(x_1, x_2, \ldots, x_n)$ is the objective function expressed in terms of these decision variables. Depending on the nature of the problem, the term *optimize* means either *maximize* or *minimize* the value of the objective function. As indicated in Equation A.2, each constraint can take the form of an equality ($=$) or an inequality ($\leq$ or $\geq$) relationship.

Complicating Factors in Optimization

Several factors can make optimization problems fairly complex and difficult to solve. One such complicating factor is the *existence of multiple decision variables* in a problem. Relatively simple procedures exist for determining the profit-maximizing output level for the single-product firm. However, the typical medium- or large-size firm often produces a large number of different products, and as a result, the profit-maximization problem for such a firm requires a series of output decisions—one for each product. Another factor that may add to the difficulty of solving a problem is the *complex nature of the relationships between the decision variables and the associated outcome.* For example, in public policy decisions on government spending for such items as education, it is extremely difficult to determine the relationship between a given expenditure and the benefits of increased income, employment, and productivity it provides. No simple relationship exists among the variables. Many of the optimization techniques discussed here are only applicable to situations in which a relatively simple function or relationship can be postulated between the decision variables and the outcome variable. A third complicating factor is the possible *existence of one or more complex constraints on the decision variables.* For example, virtually every organization has constraints imposed on its decision variables by the limited resources—such as capital, personnel, and facilities—over which it has control. These constraints must be incorporated into the decision problem. Otherwise, the optimization techniques that are applied to the problem may yield a solution that is unacceptable from a practical standpoint. Another complicating factor is the presence of *uncertainty* or *risk*. In this chapter, we limit the analysis to decision making under *certainty,* that is, problems in which each action is known to lead to a specific outcome. Chapter 2 examines methods for analyzing decisions involving risk and uncertainty. These factors illustrate the difficulties that may be encountered and may render a problem unsolvable by formal optimization procedures.

Constrained versus Unconstrained Optimization

The mathematical techniques used to solve an optimization problem represented by Equations A.1 and A.2 depend on the form of the criterion and constraint functions. The simplest situation to be considered is the *unconstrained* optimization problem. In such a problem no constraints are imposed on the decision variables, and *differential calculus* can be used to analyze them. Another relatively simple form of the general optimization problem is the case in which all the constraints of the problem can be expressed as *equality*

(=) relationships. The technique of *Lagrangian multipliers* can be used to find the optimal solution to many of these problems.

Often, however, the constraints in an economic decision-making problem take the form of *inequality* relationships (≤ or ≥) rather than equalities. For example, limitations on the resources—such as personnel and capital—of an organization place an *upper bound* or budget ceiling on the quantity of these resources that can be employed in maximizing (or minimizing) the objective function. With this type of constraint, all of a given resource need not be used in an optimal solution to the problem. An example of a *lower bound* would be a loan agreement that requires a firm to maintain a *current ratio* (that is, ratio of current assets to current liabilities) of at least 2.00. Any combination of current assets and current liabilities having a ratio greater than or equal to 2.00 would meet the provisions of the loan agreement. Such optimization procedures as the Lagrangian multiplier method are not suited to solving problems of this type efficiently; however, modern mathematical programming techniques have been developed that can efficiently solve several classes of problems with these inequality restrictions.

Linear-programming problems constitute the most important class for which efficient solution techniques have been developed. In a linear-programming problem, both the objective and the constraint relationships are expressed as linear functions of decision variables.[2] Other classes of problems include *integer-programming* problems, in which some (or all) of the decision variables are required to take on integer values, and *quadratic-programming* problems, in which the objective relationship is a quadratic function of the decision variables.[3] Generalized computing algorithms exist for solving optimization problems that meet these requirements.

The remainder of this chapter deals with the classical optimization procedures of differential calculus. Lagrangian multiplier techniques are covered in the Appendix. Linear programming is encountered in Web Chapter B.

Example

CONSTRAINED OPTIMIZATION: OPTIMIZING FLIGHT CREW SCHEDULES AT AMERICAN AIRLINES[4]

One problem that faces major airlines, such as American Airlines, is the development of a schedule for airline crews (pilots and flight attendants) that results in a high level of crew utilization. This scheduling problem is rather complex because of Federal Aviation Administration (FAA) rules designed to ensure that a crew can perform its duties without risk from fatigue. Union agreements require that flight crews receive pay for a contractually set number of hours of each day or trip. The goal for airline planners is to construct a crew schedule that meets or exceeds crew pay guarantees and does not violate FAA rules. With the salary of airline captains at $140,000 per year or more, it is important that an airline make the maximum possible usage of its crew personnel. Crew costs are the second largest direct operating cost of an airline.

The nature of the constrained optimization problem facing an airline planner is to *minimize the cost of flying the published schedule*, subject to the following constraints:

1. Each flight is assigned only one crew.

http://

Decision science modeling at American Airlines is described at http://www.amrcorp.com/corpcomm.htm

[2]A linear relationship of the variables $x_1, x_2, \ldots, x_n$ is a function of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

where all the x variables have exponents of 1.

[3]A quadratic function contains either squared terms (x_i^2) or cross-product terms (x_ix_j).

[4]Ira Gershkoff, "Optimizing Flight Crew Schedules," *Interfaces,* July/August 1989, pp. 29–43.

2. Each pairing of a crew and a flight must begin and end at a home "crew base," such as Chicago or Dallas for American.
3. Each pairing must be consistent with union work rules and FAA rules.
4. The number of jobs at each crew base must be within targeted minimum and maximum limits as specified in American's personnel plan.

American was able to develop a sophisticated constrained optimization model that saved $18 million per year compared with previous crew allocation models used.

Differential Calculus

In Chapter 2, marginal analysis was introduced as one of the fundamental concepts of economic decision making. In the marginal analysis framework, resource-allocation decisions are made by comparing the marginal benefits of a change in the level of an activity with the marginal costs of the change. A change should be made as long as the marginal benefits exceed the marginal costs. By following this basic rule, resources can be allocated efficiently and profits or shareholder wealth can be maximized.

In the profit-maximization example developed in Chapter 2, the application of the marginal analysis principles required that the relationship between the objective (profit) and the decision variable (output level) be expressed in either tabular or graphic form. This framework, however, can become cumbersome when dealing with several decision variables or with complex relationships between the decision variables and the objective. When the relationship between the decision variables and criterion can be expressed in *algebraic* form, the more powerful concepts of differential calculus can be used to find optimal solutions to these problems.

Relationship between Marginal Analysis and Differential Calculus

Initially, let us assume that the objective we are seeking to optimize, Y, can be expressed algebraically as a function of *one* decision variable, X,

$$Y = f(X) \qquad [A.3]$$

Recall that marginal profit is defined as the change in profit resulting from a one-unit change in output. In general, the marginal value of any variable Y, which is a function of another variable X, is defined as the change in the value of Y resulting from a one-unit change in X. The marginal value of Y, M_y, can be calculated from the change in Y, ΔY, that occurs as the result of a given change in X, ΔX:

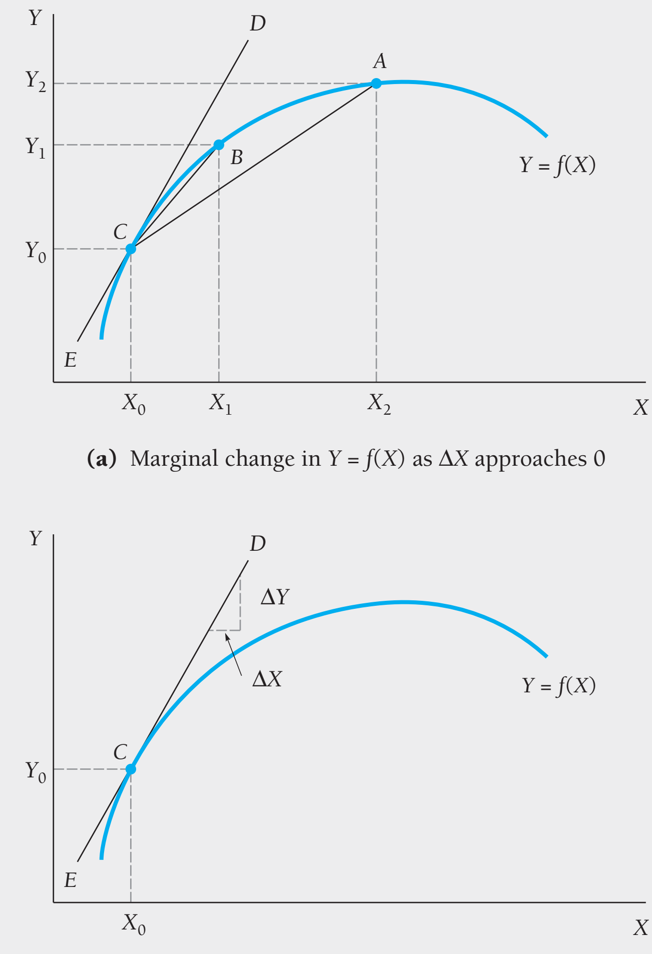
$$M_y = \frac{\Delta Y}{\Delta X} \qquad [A.4]$$

When calculated with this expression, different estimates for the marginal value of Y may be obtained, depending on the size of the change in X that we use in the computation. The true marginal value of a function (e.g., an economic relationship) is obtained from Equation A.4 when ΔX is made as small as possible. If ΔX can be thought of as a *continuous* (rather than a discrete) variable that can take on fractional values,[5] then in calculating M_y by Equation A.4, we can let ΔX approach zero. In concept, this is the approach

[5]For example, if X is a continuous variable measured in feet, pounds, and so on, then ΔX can in theory take on fractional values such as 0.5, 0.10, 0.05, 0.001, 0.0001 feet or pounds. When X is a continuous variable, ΔX can be made as small as desired.

FIGURE A.1

First Derivative of a Function



Derivative
Measures the marginal effect of a change in one variable on the value of a function. Graphically, it represents the slope of the function at a given point.

taken in differential calculus. The **derivative**, or more precisely, *first derivative*,[6] dY/dX, of a function is defined as the *limit* of the ratio $\Delta Y/\Delta X$ as ΔX approaches zero; that is,

$$\frac{dY}{dX} = \underset{\Delta X \to 0}{\text{limit}} \frac{\Delta Y}{\Delta X} \qquad [A.5]$$

Graphically, the first derivative of a function represents the *slope* of the curve at a given point on the curve. The definition of a derivative as the limit of the change in Y (that is, ΔY) as ΔX approaches zero is illustrated in Figure A.1(a). Suppose we are interested in the derivative of the $Y = f(X)$ function at the point X_0. The derivative dY/dX measures the slope of the tangent line *ECD*. An estimate of this slope, albeit a

[6]It is also possible to compute second, third, fourth, and so on, derivatives. Second derivatives are discussed later in this chapter.

poor estimate, can be obtained by calculating the marginal value of Y over the interval X_0 to X_2. Using Equation A.4, a value of

$$M'_y = \frac{\Delta Y}{\Delta X} = \frac{Y_2 - Y_0}{X_2 - X_0}$$

is obtained for the slope of the *CA* line. Now let us calculate the marginal value of Y using a smaller interval, for example, X_0 to X_1. The slope of the line C to B, which is equal to

$$M''_y = \frac{\Delta Y}{\Delta X} = \frac{Y_1 - Y_0}{X_1 - X_0}$$

gives a much better estimate of the true marginal value as represented by the slope of the *ECD* tangent line. Thus we see that the smaller the ΔX value, the better the estimate of the slope of the curve. Letting ΔX approach zero allows us to find the slope of the $Y = f(X)$ curve at point C. As shown in Figure A.1(b), the slope of the *ECD* tangent line (and the $Y = f(X)$ function at point C) is measured by the change in Y, or rise, ΔY, divided by the change in X, or run, ΔX.

Process of Differentiation

The process of differentiation—that is, finding the derivative of a function—involves determining the limiting value of the ratio $\Delta Y/\Delta X$ as ΔX approaches zero. Before offering some general rules for finding the derivative of a function, we illustrate with an example the algebraic process used to obtain the derivative without the aid of these general rules. The specific rules that simplify this process are presented in the following section.

Example

Process of Differentiation: Profit Maximization at Illinois Power

Suppose the profit, π, of Illinois Power can be represented as a function of the output level Q using the expression

$$\pi = -40 + 140Q - 10Q^2 \quad \text{[A.6]}$$

We wish to determine $d\pi/dQ$ by first finding the marginal-profit expression $\Delta\pi/\Delta Q$ and then taking the limit of this expression as ΔQ approaches zero. Let us begin by expressing the new level of profit $(\pi + \Delta\pi)$ that will result from an increase in output to $(Q + \Delta Q)$. From Equation A.6, we know that

$$(\pi + \Delta\pi) = -40 + 140(Q + \Delta Q) - 10(Q + \Delta Q)^2 \quad \text{[A.7]}$$

Expanding this expression and then doing some algebraic simplifying, we obtain

$$\begin{aligned}(\pi + \Delta\pi) &= -40 + 140Q + 140\Delta Q - 10[Q^2 + 2Q\Delta Q + (\Delta Q)^2] \\ &= -40 + 140Q - 10Q^2 + 140\Delta Q - 20Q\Delta Q - 10(\Delta Q)^2 \quad \text{[A.8]}\end{aligned}$$

Subtracting Equation A.6 from Equation A.8 yields

$$\Delta\pi = 140\Delta Q - 20Q\Delta Q - 10(\Delta Q)^2 \quad \text{[A.9]}$$

Forming the marginal-profit ratio $\Delta\pi/\Delta Q$, and doing some canceling, we get

$$\begin{aligned}\frac{\Delta\pi}{\Delta Q} &= \frac{140\Delta Q - 20Q\Delta Q - 10(\Delta Q)^2}{\Delta Q} \quad \text{[A.10]} \\ &= 140 - 20Q - 10\Delta Q\end{aligned}$$

http://
You can connect to Illinois Power Company, a subsidiary of Illinova, at http://www.illinova.com/ Clicking on the "inside Illinova" button will give you access to financial information.

Taking the limit of Equation A.10 as ΔQ approaches zero yields the expression for the derivative of Illinois Power's profit function (Equation A.6)

$$\frac{d\pi}{dQ} = \lim_{\Delta Q \to 0} [140 - 20Q - 10\Delta Q] = 140 - 20Q \qquad [A.11]$$

If we are interested in the derivative of the profit function at a particular value of Q, Equation A.11 can be evaluated for this value. For example, suppose we want to know the marginal profit, or slope of the profit function, at $Q = 3$ units. Substituting $Q = 3$ in Equation A.11 yields

$$\text{Marginal profit} = \frac{d\pi}{dQ} = 140 - 20(3) = \$80 \text{ per unit}$$

Rules of Differentiation

Fortunately, we do not need to go through this lengthy process every time we want the derivative of a function. A series of general rules, derived in a manner similar to the process just described, exists for differentiating various types of functions.[7]

Constant Functions A constant function can be expressed as

$$Y = a \qquad [A.12]$$

where a is a constant (that is, Y is independent of X). The derivative of a constant function is equal to zero:

$$\frac{dY}{dX} = 0 \qquad [A.13]$$

For example, consider the constant function

$$Y = 4$$

which is graphed in Figure A.2(a). Recall that the first derivative of a function (dY/dX) measures the slope of the function. Because this constant function is a horizontal straight line with zero slope, its derivative (dY/dX) is therefore equal to zero.

Power Functions A power function takes the form of

$$Y = aX^b \qquad [A.14]$$

where a and b are constants. The derivative of a power function is equal to b times a, times X raised to the $(b - 1)$ power:

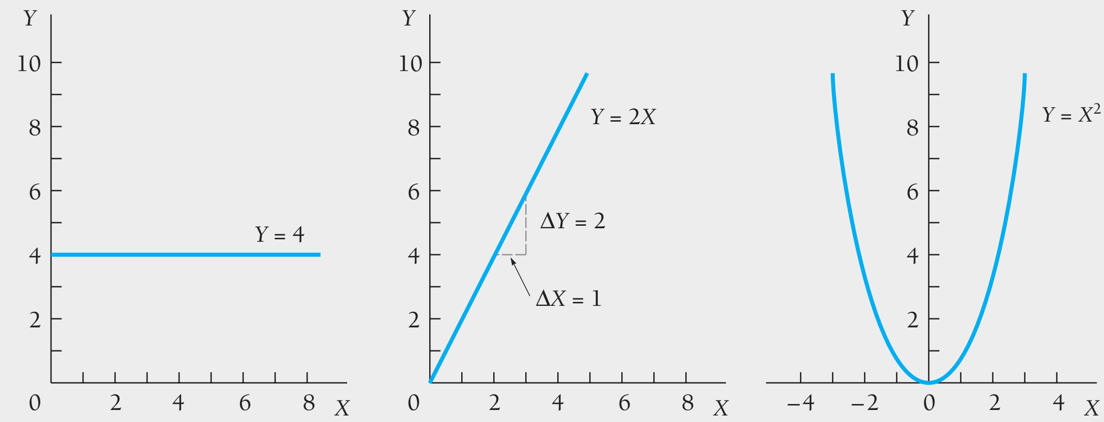
$$\frac{dY}{dX} = b \cdot a \cdot X^{b-1} \qquad [A.15]$$

A couple of examples are used to illustrate the application of this rule. First, consider the function

$$Y = 2X$$

[7]A more expanded treatment of these rules can be found in any introductory calculus book such as André L. Yandl, *Applied Calculus* (Belmont, Calif.: Wadsworth, 1991).

FIGURE A.2 Constant, Linear, and Quadratic Functions



(a) Constant function **(b)** Linear function **(c)** Quadratic function

which is graphed in Figure A.2(b). Note that the slope of this function is equal to 2 and is constant over the entire range of X values. Applying the power function rule to this example, where $a = 2$ and $b = 1$, yields

$$\frac{dY}{dX} = 1 \cdot 2 \cdot X^{1-1} = 2X^0$$
$$= 2$$

Note that any variable to the zero power, e.g., X^0, is equal to 1.

Next, consider the function

$$Y = X^2$$

which is graphed in Figure A.2(c). Note that the slope of this function varies depending on the value of X. Application of the power function rule to this example yields ($a = 1, b = 2$):

$$\frac{dY}{dX} = 2 \cdot 1 \cdot X^{2-1}$$
$$= 2X$$

As we can see, this derivative (or slope) function is negative when $X < 0$, zero when $X = 0$, and positive when $X > 0$.

Sums of Functions Suppose a function $Y = f(X)$ represents the sum of two (or more) separate functions, $f_1(X), f_2(X)$, that is,

$$Y = f_1(X) + f_2(X) \qquad [A.16]$$

The derivative of Y with respect to X is found by differentiating each of the separate functions and then adding the results:

$$\frac{dY}{dX} = \frac{df_1(X)}{dX} + \frac{df_2(X)}{dX} \qquad [A.17]$$

This result can be extended to finding the derivative of the sum of any number of functions.

Example

Rules of Differentiation: Profit Maximization at Illinois Power (continued)

As an example of the application of these rules, consider again the profit function for Illinois Power, given by Equation A.6, that was discussed earlier:

$$\pi = -40 + 140Q - 10Q^2$$

In this example Q represents the X variable and π represents the Y variable; that is, $\pi = f(Q)$. The function $f(Q)$ is the sum of *three* separate functions—a constant function, $f_1(Q) = -40$, and two power functions, $f_2(Q) = 140Q$ and $f_3(Q) = -10Q^2$. Therefore, applying the differentiation rules yields

$$\begin{aligned} \frac{d\pi}{dQ} &= \frac{df_1(Q)}{dQ} + \frac{df_2(Q)}{dQ} + \frac{df_3(Q)}{dQ} \\ &= 0 + 1 \cdot 140 \cdot Q^{1-1} + 2 \cdot (-10) \cdot Q^{2-1} \\ &= 140 - 20Q \end{aligned}$$

This is the same result that was obtained earlier in Equation A.11 by the differentiation process.

Product of Two Functions Suppose the variable Y is equal to the product of two separate functions $f_1(X)$ and $f_2(X)$:

$$Y = f_1(X) \cdot f_2(X) \qquad [A.18]$$

In this case the derivative of Y with respect to X is equal to the sum of the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{dY}{dX} = f_1(X) \cdot \frac{df_2(X)}{dX} + f_2(X) \cdot \frac{df_1(X)}{dX} \qquad [A.19]$$

For example, suppose we are interested in the derivative of the expression

$$Y = X^2(2X - 3)$$

Let $f_1(X) = X^2$ and $f_2(X) = (2X - 3)$. By the above rule (and the earlier rules for differentiating constant and power functions), we obtain

$$\begin{aligned} \frac{dY}{dX} &= X^2 \cdot \frac{d}{dX}[(2X-3)] + (2X-3) \cdot \frac{d}{dX}[X^2] \\ &= X^2 \cdot (2 - 0) + (2X - 3) \cdot (2X) \\ &= 2X^2 + 4X^2 - 6X \\ &= 6X^2 - 6X \\ &= 6X(X - 1) \end{aligned}$$

Quotient of Two Functions Suppose the variable Y is equal to the quotient of two separate functions $f_1(X)$ and $f_2(X)$:

$$Y = \frac{f_1(X)}{f_2(X)} \qquad [A.20]$$

For such a relationship the derivative of Y with respect to X is obtained as follows:

$$\frac{dY}{dX} = \frac{f_2(X) \cdot \frac{df_1(X)}{dX} - f_1(X) \cdot \frac{df_2(X)}{dX}}{[f_2(X)]^2} \qquad [A.21]$$

As an example, consider the problem of finding the derivative of the expression

$$Y = \frac{10X^2}{5X - 1}$$

Letting $f_1(X) = 10X^2$ and $f_2(X) = 5X - 1$, we have

$$\begin{aligned}\frac{dY}{dX} &= \frac{(5X - 1) \cdot 20X - 10X^2 \cdot 5}{(5X - 1)^2} \\ &= \frac{100X^2 - 20X - 50X^2}{(5X - 1)^2} \\ &= \frac{50X^2 - 20X}{(5X - 1)^2} \\ &= \frac{10X(5X - 2)}{(5X - 1)^2}\end{aligned}$$

Functions of a Function (Chain Rule) Suppose Y is a function of the variable Z, $Y = f_1(Z)$; and Z is in turn a function of the variable X, $Z = f_2(X)$. The derivative of Y with respect to X can be determined by first finding dY/dZ and dZ/dX and then multiplying the two expressions together:

$$\begin{aligned}\frac{dY}{dX} &= \frac{dY}{dZ} \cdot \frac{dZ}{dX} \\ &= \frac{df_1(Z)}{dZ} \cdot \frac{df_2(X)}{dX}\end{aligned} \qquad [A.22]$$

To illustrate the application of this rule, suppose we are interested in finding the derivative (with respect to X) of the function

$$Y = 10Z - 2Z^2 - 3$$

where Z is related to X in the following way:[8]

$$Z = 2X^2 - 1$$

First, we find (by the earlier differentiation rules)

$$\frac{dY}{dZ} = 10 - 4Z$$

$$\frac{dZ}{dX} = 4X$$

and then

$$\frac{dY}{dX} = (10 - 4Z) \cdot 4X$$

[8]Alternatively, one can substitute $Z = 2X^2 - 1$ into $Y = 10Z - 2Z^2 - 3$ and differentiate Y with respect to X. The reader is asked to demonstrate in Exercise 24 that this approach yields the same answer as the chain rule.

TABLE A.1

Summary of Rules for Differentiating Functions

Function	Derivative
1. Constant Function	
$Y = a$	$\frac{dY}{dX} = 0$
2. Power Function	
$Y = aX^b$	$\frac{dY}{dX} = b \cdot a \cdot X^{b-1}$
3. Sums of Functions	
$Y = f_1(X) + f_2(X)$	$\frac{dY}{dX} = \frac{df_1(X)}{dX} + \frac{df_2(X)}{dX}$
4. Product of Two Functions	
$Y = f_1(X) \cdot f_2(X)$	$\frac{dY}{dX} = f_1(X) \cdot \frac{df_2(X)}{dX} + f_2(X) \cdot \frac{df_1(X)}{dX}$
5. Quotient of Two Functions	
$Y = \frac{f_1(X)}{f_2(X)}$	$\frac{dY}{dX} = \frac{f_2(X) \cdot \frac{df_1(X)}{dX} - f_1(X) \cdot \frac{df_2(X)}{dX}}{[f_2(X)]^2}$
6. Functions of a Function	
$Y = f_1(Z)$, where $Z = f_2(X)$	$\frac{dY}{dX} = \frac{dY}{dZ} \cdot \frac{dZ}{dX}$

Substituting the expression for Z in terms of X into this equation yields

$$\begin{aligned}\frac{dY}{dX} &= [10 - 4(2X^2 - 1)] \cdot 4X \\ &= (10 - 8X^2 + 4) \cdot 4X \\ &= 40X - 32X^3 + 16X \\ &= 56X - 32X^3 \\ &= 8X(7 - 4X^2)\end{aligned}$$

These rules for differentiating functions are summarized in Table A.1.

Applications of Differential Calculus to Optimization Problems

The reason for studying the process of differentiation and the rules for differentiating functions is that these methods can be used to find optimal solutions to many kinds of maximization and minimization problems in managerial economics.

Maximization Problem

First-Order Condition A test to locate one or more maximum or minimum points of an algebraic function.

Recall from the discussion of marginal analysis, a necessary (but not sufficient) condition for finding the maximum point on a curve (for example, maximum profits) is that the marginal value or slope of the curve at this point must be equal to zero. We can now express this condition within the framework of differential calculus. Because the derivative of a function measures the slope or marginal value at any given point, an equivalent necessary condition for finding the maximum value of a function $Y = f(X)$ is that the derivative dY/dX at this point must be equal to zero. This is known as the **first-order condition** for locating one or more maximum or minimum points of an algebraic function.

Example

First-Order Condition: Profit Maximization at Illinois Power (continued)

Using the profit function (Equation A.6)

$$\pi = -40 + 140Q - 10Q^2$$

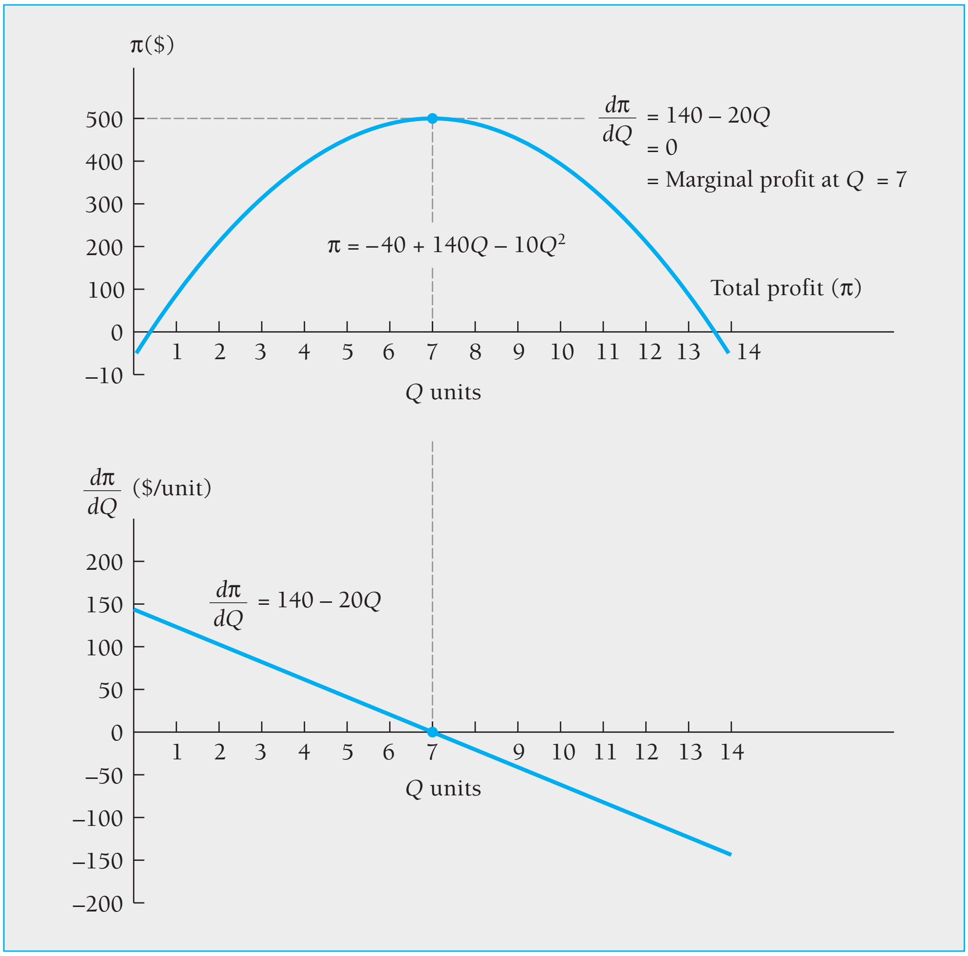
discussed earlier, we can illustrate how to find the profit-maximizing output level Q by means of this condition. Setting the first derivative of this function (which was computed previously) to zero, we obtain

$$\frac{d\pi}{dQ} = 140 - 20Q$$
$$0 = 140 - 20Q$$

Solving this equation for Q yields $Q^* = 7$ units as the profit-maximizing output level. The profit and first derivative functions and optimal solution are shown in Figure A.3. As we can see, profits are maximized at the point where the function is neither increasing nor decreasing; in other words, where the slope (or first derivative) is equal to zero.

FIGURE A.3

Profit and First Derivative Functions



Second Derivatives and the Second-Order Condition

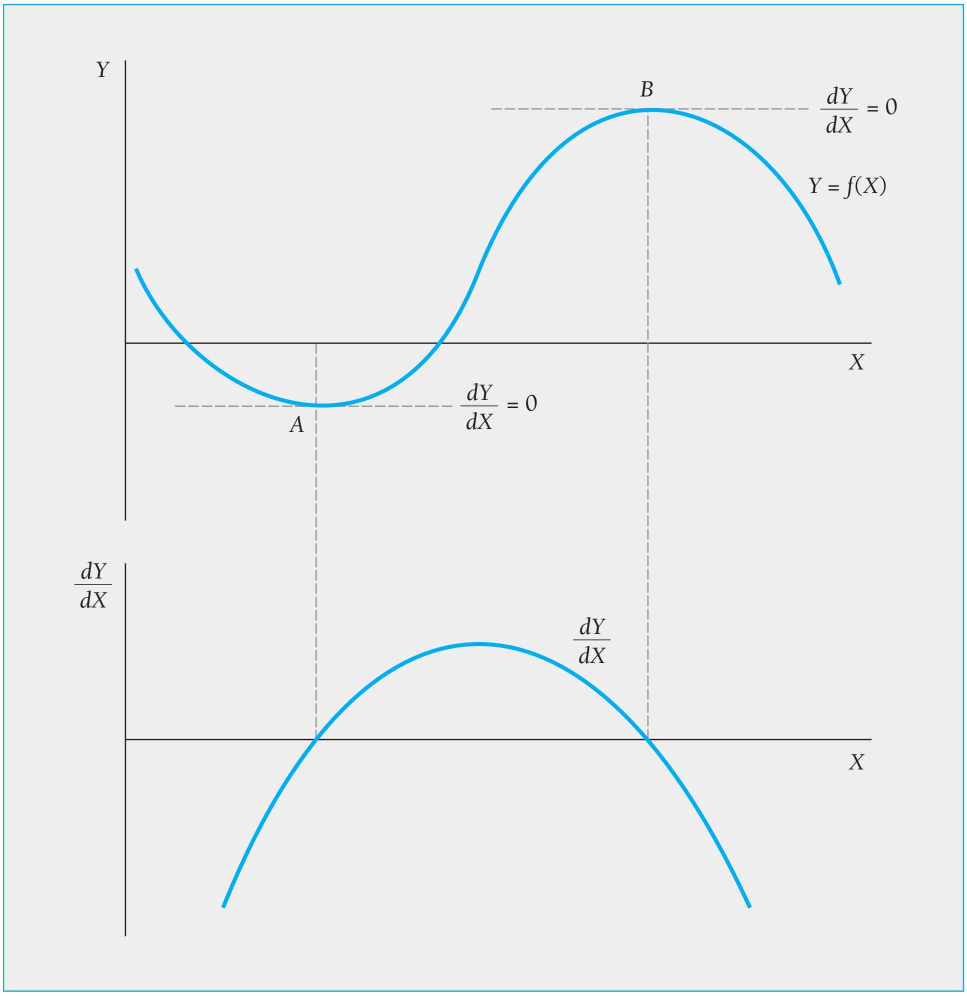
Second-Order Condition A test to determine whether a point that has been determined from the first-order condition is either a maximum point or a minimum point of the algebraic function.

Setting the derivative of a function equal to zero and solving the resulting equation for the value of the decision variable does not guarantee that the point will be obtained at which the function takes on its maximum value. (Recall the Stealth bomber example at the start of the chapter.) The slope of a U-shaped function will also be equal to zero at its low point and the function will take on its *minimum* value at the given point. In other words, setting the derivative to zero is only a *necessary* condition for finding the maximum value of a function; it is not a *sufficient* condition. Another condition, known as the **second-order condition**, is required to determine whether a point that has been determined from the first-order condition is either a maximum point or minimum point of the algebraic function.

This situation is illustrated in Figure A.4. At both points A and B the slope of the function (first derivative, dY/dX) is zero; however, only at point B does the function take on its maximum value. We note in Figure A.4 that the marginal value (slope) is continually *decreasing* in the neighborhood of the maximum value (point B) of the $Y = f(X)$ function. First the slope is positive up to the point where $dY/dX = 0$, and thereafter the slope becomes negative. Thus we must determine whether the slope's marginal value (slope of

FIGURE A.4

Maximum and Minimum Values of a Function



the slope) is declining. A test to see whether the marginal value is decreasing is to take the derivative of the marginal value and check to see if it is negative at the given point on the function. In effect, we need to find the derivative of the derivative—that is, the *second derivative* of the function—and then test to see if it is less than zero. Formally, the second derivative of the function $Y = f(X)$ is written as d^2Y/dX^2 and is found by applying the previously described differentiation rules to the first derivative. A *maximum point is obtained if the second derivative is negative; that is,* $d^2Y/dX^2 < 0$.

Example

SECOND-ORDER CONDITION: PROFIT MAXIMIZATION AT ILLINOIS POWER (CONTINUED)

Returning to the profit-maximization example, the second derivative is obtained from the first derivative as follows:

$$\frac{d\pi}{dQ} = 140 - 20Q$$

$$\frac{d^2\pi}{dQ^2} = 0 + 1 \cdot (-20) \cdot Q^{1-1}$$

$$= -20$$

Because $d^2\pi/dQ^2 < 0$, we know that a maximum-profit point has been obtained.

An opposite condition holds for obtaining the point at which the function takes on a minimum value. Note again in Figure A.4 that the marginal value (slope) is continually *increasing* in the neighborhood of the minimum value (point A) of the $Y = (X)$ function. First the slope is negative up to the point where $dY/dX = 0$, and thereafter the slope becomes positive. Therefore, we test to see if $d^2Y/dX^2 > 0$ at the given point. A *minimum point is obtained if the second derivative is positive; that is,* $d^2Y/dX^2 > 0$.

Minimization Problem

In some decision-making situations, cost minimization may be the objective. As in profit-maximization problems, differential calculus can be used to locate the optimal points.

Example

COST MINIMIZATION: KEYSPAN ENERGY

Suppose we are interested in determining the output level that minimizes average total costs for KeySpan Energy, where the average total cost function might be approximated by the following relationship (Q represents output):

$$C = 15 - .040Q + .000080Q^2 \quad [A.23]$$

Differentiating C with respect to Q gives

$$\frac{dC}{dQ} = -.040 + .000160Q$$

Setting this derivative equal to zero and solving for Q yields

$$0 = -.040 + .000160Q$$

$$Q^* = 250$$

http://

KeySpan Energy is a natural gas distribution company. You can access financial information on KeySpan Energy at their Internet site: http://www.keyspanenergy.com

Taking the second derivative, we obtain

$$\frac{d^2C}{dQ^2} = +.000160$$

Because the second derivative is positive, the output level of $Q = 250$ is indeed the value that minimizes average total costs.

Summarizing, we see that *two* conditions are required for locating a maximum or minimum value of a function using differential calculus. The *first-order* condition determines the point(s) at which the first derivative dY/dX is equal to zero. Having obtained one or more points, a *second-order* condition is used to determine whether the function takes on a maximum or minimum value at the given point(s). The second derivative d^2Y/dX^2 indicates whether a given point is a maximum ($d^2Y/dX^2 < 0$) or a minimum ($d^2Y/dX^2 > 0$) value of the function.

Partial Differentiation and Multivariate Optimization

Thus far in the chapter, the analysis has been limited to a criterion variable Y that can be expressed as a function of *one* decision variable X. However, many commonly used economic relationships contain two or more decision variables. For example, a *production function* relates the output of a plant, firm, industry, or country to the inputs employed—such as capital, labor, and raw materials. Another example is a *demand function*, which relates sales of a product or service to such variables as price, advertising, promotion expenses, price of substitutes, and income.

Partial Derivatives

Consider a criterion variable Y that is a function of two decision variables X_1 and X_2:[9]

$$Y = f(X_1, X_2)$$

Partial Derivative Measures the marginal effect of a change in one variable on the value of a multivariate function, while holding constant all other variables.

Let us now examine the change in Y that results from a given change in either X_1 or X_2. To isolate the marginal effect on Y from a given change in X_1—that is, $\Delta Y/\Delta X_1$—we must hold X_2 constant. Similarly, if we wish to isolate the marginal effect on Y from a given change in X_2—that is, $\Delta Y/\Delta X_2$—the variable X_1 must be held constant. A measure of the marginal effect of a change in any one variable on the change in Y, holding all other variables in the relationship constant, is obtained from the **partial derivative** of the function. The partial derivative of Y with respect to X_1 is written as $\partial Y/\partial X_1$ and is found by applying the previously described differentiation rules to the $Y = f(X_1, X_2)$ function, where the variable X_2 is treated as a constant. Similarly, the partial derivative of Y with respect to X_2 is written as $\partial Y/\partial X_2$ and is found by applying the differentiation rules to the function, where the variable X_1 is treated as a constant.

[9] The following analysis is not limited to two decision variables. Relationships containing any number of variables can be analyzed within this framework.

Example

Partial Derivatives: Indiana Petroleum Company

To illustrate the procedure for obtaining partial derivatives, let us consider the following relationship in which the profit variable, π, is a function of the output level of *two* products (heating oil and gasoline) Q_1 and Q_2:

$$\pi = -60 + 140Q_1 + 100Q_2 - 10Q_1^2 - 8Q_2^2 - 6Q_1Q_2 \quad [A.24]$$

Treating Q_2 as a constant, the partial derivative of π with respect to Q_1 is obtained:

$$\begin{aligned}\frac{\partial \pi}{\partial Q_1} &= 0 + 140 + 0 + 2 \cdot (-10) \cdot Q_1 - 0 - 6Q_2 \\ &= 140 - 20Q_1 - 6Q_2\end{aligned} \quad [A.25]$$

Similarly, with Q_1 treated as a constant, the partial derivative of π with respect to Q_2 is equal to

$$\begin{aligned}\frac{\partial \pi}{\partial Q_2} &= 0 + 0 + 100 - 0 + 2 \cdot (-8) \cdot Q_2 - 6Q_1 \\ &= 100 - 16Q_2 - 6Q_1\end{aligned} \quad [A.26]$$

Example

Partial Derivatives: Demand Function for Shield Toothpaste

As another example, consider the following (multiplicative) demand function, where Q = quantity sold, P = selling price, and A = advertising expenditures:

$$Q = 3.0P^{-.50}A^{.25} \quad [A.27]$$

The partial derivative of Q with respect to P is

$$\begin{aligned}\frac{\partial Q}{\partial P} &= 3.0A^{.25}(-.50P^{-.50-1}) \\ &= -1.5P^{-1.50}A^{.25}\end{aligned}$$

Similarly, the partial derivative of Q with respect to A is

$$\begin{aligned}\frac{\partial Q}{\partial A} &= 3.0P^{-.50}(.25A^{.25-1}) \\ &= .75P^{-.50}A^{-.75}\end{aligned}$$

Maximization Problem

The partial derivatives can be used to obtain the optimal solution to a maximization or minimization problem containing two or more X variables. Analogous to the first-order conditions discussed earlier for the one-variable case, we set *each* of the partial derivatives equal to zero and solve the resulting set of simultaneous equations for the optimal X values.

Example

Profit Maximization: Indiana Petroleum Company (continued)

Suppose we are interested in determining the values of Q_1 and Q_2 that maximize the company's profits given in Equation A.24. In this case, each of the two partial derivative functions (Equations A.25 and A.26) would be set equal to zero:

$$0 = 140 - 20Q_1 - 6Q_2 \quad [A.28]$$

$$0 = 100 - 16Q_2 - 6Q_1 \quad [A.29]$$

This system of equations can be solved for the profit-maximizing values of Q_1 and Q_2.[10] The optimal values are $Q_1^* = 5.77$ units and $Q_2^* = 4.08$ units.[11] The optimal total profit is equal to

$$\pi^* = -60 + 140(5.77) + 100(4.08) - 10(5.77)^2 - 8(4.08)^2 - 6(5.77)(4.08) = 548.45$$

International Perspectives

Dealing with Import Restraints: Toyota

During the 1992 U.S. presidential campaign, there was extensive rhetoric about the "problem" of the U.S. balance of trade deficit with Japan, particularly about the level of Japanese auto imports into the United States. Some of the proposals that have been advanced to reduce the magnitude of this "problem," and thereby assist the U.S. auto industry, include the imposition of rigid car import quotas. Japanese manufacturers would be forced to restrict the number of cars that are exported to the United States.

Had rigid import quotas been imposed, Japanese manufacturers would have had to take this constraint into consideration when making production, distribution, and new car introduction plans. Japanese manufacturers could no longer just seek to maximize profits. Profits could only be maximized subject to an aggregate constrained level of exports that could be sent to the U.S. market.

Shortly after, Japanese manufacturers responded to import constraints by raising prices, thereby making Japanese cars less price competitive with U.S.-built autos. Increases in prices by Japanese firms would give U.S. manufacturers additional room for raising their own prices. Indeed, in early 1992 in the face of threatened import quotas, Toyota announced significant price increases on the cars it sold in the United States. Over the longer term, Japanese firms have shifted their product mix to more profitable, larger (luxury) cars, abandoning some of their market share in smaller, less profitable vehicles to U.S. manufacturers.

As can be seen in this example, the imposition of additional constraints on the operating activities of a firm can have a substantial impact on its short- and long-term pricing and output strategies.

http://

You can read more about trade tariffs and quotas at the Internet site maintained by the National Center for Policy Analysis: http://www.public-policy.org/~ncpa/studies/s171/s171.html

Summary

- Within the area of decision-making under certainty are two broad classes of problems—*unconstrained* optimization problems and *constrained* optimization problems.
- *Marginal analysis* is useful in making decisions about the expansion or contraction of an economic activity.
- *Differential calculus,* which bears a close relationship to marginal analysis, can be applied whenever an algebraic relationship can be specified between the decision variables and the objective or criterion variable.
- The *first-derivative* measures the slope or rate of change of a function at a given point and is equal to the limiting value of the marginal function as the marginal value is calculated over smaller and smaller intervals, that is, as the interval approaches zero.

---

[10] The second-order conditions for obtaining a maximum or minimum in the multiple-variable case are somewhat complex. A discussion of these conditions can be found in most basic calculus texts.

[11] Exercise 25 at the end of the chapter requires the determination of these optimal values.

- Various rules are available (see Table A.1) for finding the derivative of specific types of functions.
- A necessary, but not sufficient, condition for finding the maximum or minimum points of a function is that the first derivative be equal to zero. This is known as the *first-order condition*.
- A *second-order condition* is required to determine whether a given point is a maximum or minimum. The *second derivative* indicates that a given point is a maximum if the second derivative is less than zero or a minimum if the second derivative is greater than zero.
- The *partial derivative* of a multivariate function measures the marginal effect of a change in one variable on the value of the function, holding constant all other variables.
- In constrained optimization problems, *Lagrangian multiplier techniques* can be used to find the optimal value of a function that is subject to *equality* constraints. Through the introduction of additional (artificial) variables into the problem, the Lagrangian multiplier method converts the constrained problem into an unconstrained problem, which can then be solved using ordinary differential calculus procedures. Lagrangian multiplier techniques are discussed and illustrated in the Appendix to this chapter.

EXERCISES

1. Explain how the first and second derivatives of a function are used to find the maximum or minimum points of a function $Y = f(X)$. Illustrate your discussion with graphs.

2. Why is the first-order condition for finding a maximum (or minimum) of a function referred to as a necessary, but not sufficient, condition?

3. Defining Q to be the level of output produced and sold, suppose that the firm's total revenue (TR) and total cost (TC) functions can be represented in tabular form as shown below.

Output *Q*	Total Revenue *TR*	Total Cost *TC*	Output *Q*	Total Revenue *TR*	Total Cost *TC*
0	0	20	11	264	196
1	34	26	12	276	224
2	66	34	13	286	254
3	96	44	14	294	286
4	124	56	15	300	320
5	150	70	16	304	356
6	174	86	17	306	394
7	196	104	18	306	434
8	216	124	19	304	476
9	234	146	20	300	520
10	250	170			

a. Compute the marginal revenue and average revenue functions.
b. Compute the marginal cost and average cost functions.

- **c.** On a single graph, plot the total revenue, total cost, marginal revenue, and marginal cost functions.
- **d.** Determine the output level in the *graph* that maximizes profits (that is, profit = total revenue − total cost) by finding the point where marginal revenue equals marginal cost.
- **e.** Check your result in part (d) by finding the output level in the *tables* developed in parts (a) and (b) that likewise satisfies the condition that marginal revenue equals marginal cost.

4. Consider again the total revenue and total cost functions shown in tabular form in the previous problem.

- **a.** Compute the total, marginal, and average profit functions.
- **b.** On a single graph, plot the total profit and marginal profit functions.
- **c.** Determine the output level in the graph and table where the total profit function takes on its maximum value.
- **d.** How does the result in part (c) in this exercise compare with the result in part (d) of the previous exercise?
- **e.** Determine total profits at the profit-maximizing output level.

5. Differentiate the following functions:

- **a.** $TC = 50 + 100Q - 6Q^2 + .5Q^3$
- **b.** $ATC = 50/Q + 100 - 6Q + .5Q^2$
- **c.** $MC = 100 - 12Q + 1.5Q^2$
- **d.** $Q = 50 - .75P$
- **e.** $Q = .40X^{1.50}$

6. Differentiate the following functions:

- **a.** $Y = 2X^3/(4X^2 - 1)$
- **b.** $Y = 2X^3(4X^2 - 1)$
- **c.** $Y = 8Z^2 - 4Z + 1$, where $Z = 2X^2 - 1$ (differentiate Y with respect to X)

7. Defining Q to be the level of output produced and sold, assume that the firm's cost function is given by the relationship

$$TC = 20 + 5Q + Q^2$$

Furthermore, assume that the demand for the output of the firm is a function of price P given by the relationship

$$Q = 25 - P$$

- **a.** Defining total profit as the difference between total revenue and total cost, express in terms of Q the total profit function for the firm. (*Note:* Total revenue equals price per unit times the number of units sold.)
- **b.** Determine the output level where total profits are maximized.
- **c.** Calculate total profits and selling price at the profit-maximizing output level.
- **d.** If fixed costs increase from \$20 to \$25 in the total cost relationship, determine the effects of such an increase on the profit-maximizing output level and total profits.

8. Using the cost and demand functions in Exercise 7:

- **a.** Determine the marginal revenue and marginal cost functions.
- **b.** Show that, at the profit-maximizing output level determined in part (b) of the previous exercise, marginal revenue equals marginal cost. This illustrates the economic principle that profits are maximized at the output level where marginal revenue equals marginal cost.

9. Using the cost and demand functions in Exercise 7, suppose the government imposes a 20 percent *tax on the net profits* (that is, a tax on the difference between revenues and costs) of the firm.

- **a.** Determine the new profit function for the firm.
- **b.** Determine the output level at which total profits are maximized.
- **c.** Calculate total profits (after taxes) and the selling price at the profit-maximizing output level.
- **d.** Compare the results in parts (b) and (c) with the results in Exercise 7 above.

10. Suppose the government imposes a 20 percent *sales tax* (that is, a tax on revenue) on the output of the firm. Answer questions (a), (b), (c), and (d) of the previous exercise according to this new condition.

11. The Bowden Corporation's average variable cost function is given by the following relationship (where Q is the number of units produced and sold):

$$AVC = 25{,}000 - 180Q + .50Q^2$$

- **a.** Determine the output level (Q) that minimizes average variable cost.
- **b.** How does one know that the value of Q determined in part (a) *minimizes* rather than *maximizes AVC?*

12. Determine the partial derivatives with respect to all of the variables in the following functions:

- **a.** $TC = 50 + 5Q_1 + 10\,Q_2 + .5Q_1Q_2$
- **b.** $Q = 1.5L^{.60}\,K^{.50}$
- **c.** $Q_A = 2.5P_A{}^{-1.30}Y^{.20}P_B^{.40}$

13. Bounds Inc. has determined through regression analysis that its sales (S) are a function of the amount of advertising (measured in units) in two different media. This is given by the following relationship (X = newspapers, Y = magazines):

$$S(X,Y) = 200X + 100Y - 10X^2 - 20Y^2 + 20XY$$

- **a.** Find the level of newspaper and magazine advertising that maximizes the firm's sales.
- **b.** Calculate the firm's sales at the optimal values of newspaper and magazine advertising determined in part (a).

14. The Santa Fe Cookie Factory is considering an expansion of its retail piñon cookie business to other cities. The firm's owners lack the funds needed to undertake the expansion on their own. They are considering a franchise arrangement for the new outlets. The company incurs variable costs of $6 for each pound of cookies sold. The fixed costs of operating a typical retail outlet are estimated to be $300,000 per year. The demand function facing each retail outlet is estimated to be

$$P = \$50 - .001Q$$

where P is the price per pound of cookies and Q is the number of pounds of cookies sold. [*Note:* Total revenue equals price (P) times quantity (Q) sold.]

- **a.** What price, output, total revenue, total cost, and total profit level will each profit-maximizing franchise experience?
- **b.** Assuming that the parent company charges each franchisee a fee equal to 5 percent of total revenues, recompute the values in part (a) above.
- **c.** The Santa Fe Cookie Factory is considering a combined fixed/variable franchise fee structure. Under this arrangement each franchisee would pay the parent company $25,000 plus 1 percent of total revenues. Recompute the values in part (a) above.

d. What franchise fee arrangement do you recommend that the Santa Fe Cookie Factory adopt? What are the advantages and disadvantages of each plan?

15. Several fast-food chains, including McDonald's, announced that they have shifted from the use of beef tallow, which is high in saturated fat, to unsaturated vegetable fat. This change was made in response to increasing consumer awareness of the relationship between food and health.
   a. Why do you believe McDonald's used beef tallow prior to the change?
   b. What impact would you expect the change to have on McDonald's profits (i) in the near term, (ii) in the long term?
   c. Structure this situation as a constrained-optimization problem. What is the objective function? What are the constraints?

16. Differentiate the following functions:
   a. $TR = 50Q - 4Q^2$
   b. $VC = 75Q - 5Q^2 + .25Q^3$
   c. $MC = 75 - 10Q + .75Q^2$
   d. $Q = 50 - 4P$
   e. $Q = 2.0\,L^{.75}$

17. Determine the marginal cost function by differentiating the following total cost function with respect to Q (output):

$$TC = a + bQ + cQ^2 + dQ^3$$

where a, b, c, and d, are constants.

18. Differentiate the following functions:
   a. $Y = \frac{1}{3}X^3/(\frac{1}{2}X^2 - 1)$
   b. $Y = \frac{1}{3}X^3(\frac{1}{2}X^2 - 1)$
   c. $Y = 2Z^2 + 2Z + 3$, where $Z = X^2 - 2$ (differentiate Y with respect to X)

19. Determine the partial derivatives with respect to all the variables in the following functions:
   a. $Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$, where α, β_1, β_2, and β_3 are constants
   b. $Q = \alpha L^{\beta_1} K^{\beta_2}$, where α, β_1, and β_2 are constants
   c. $Q_A = aP_A^b Y^c P_B^d$, where a, b, c, and d are constants

20. Given the following total revenue function (where Q = output):

$$TR = 100\,Q - 2Q^2$$

   a. Determine the level of output that maximizes revenues.
   b. Show that the value of Q determined in part (a) maximizes, rather then minimizes, revenues.

21. Given the following total profit function (where Q = output):

$$\pi = -250{,}000 + 20{,}000\,Q - 2Q^2$$

   a. Determine the level of output that maximizes profits.
   b. Show that the value of Q determined in part (a) maximizes, rather then minimizes, profits.

22. Given the following average total cost function (where Q = output):

$$ATC = 5{,}000 - 100Q + 1.0Q^2$$

   a. Determine the level of output that minimizes average total costs.
   b. Show that the value of Q determined in part (a) minimizes, rather then maximizes, average total costs.

23. Determine the value(s) of X that maximize or minimize the following functions and indicate whether each value represents either a maximum or minimum point of the function. (Hint: The roots of the quadratic equation: $aX^2 + bX + c = 0$ are $X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.)

a. $Y = \frac{1}{3}X^3 - 60X^2 + 2{,}000X + 50{,}000$

b. $Y = -\frac{1}{3}X^3 + 60X^2 - 2{,}000X + 50{,}000$

24. Show that substituting

$$Z = 2X^2 - 1$$

into

$$Y = 10Z - 2Z^2 - 3$$

and differentiating Y with respect to X yields the same result as application of the chain rule $[dY/dX = 8X(7 - 4X^2)]$.

25. Show that the optimal solution to the set of simultaneous equations in the Indiana Petroleum example, Equations A.28 and A.29, are $Q_1^* = 5.77$ and $Q_2^* = 4.08$.

http://

Import Quotas as an Optimization Constraint

26. As suggested in the text, import restrictions such as tariffs and quotas serve as constraints on the profit-maximization objective of importers. While international free trade agreements such as the North American Free Trade Agreement have generally reduced or eliminated tariffs and quotas, some still remain. One of the import quotas still in force in the United States applies to textiles. You can access information on textile quotas from several sites on the Internet, including the World Trade Organization's website at http://www.wto.org/english/tratop_e/tratop_e.htm and the Cato Institute website at http://www. cato.org/pubs/pas/pa-140es.html and write a two-paragraph executive summary of how this import quota modifies the profit-maximizing choices of both domestic and foreign textile manufacturers.

Web Chapter A Appendix Constrained Optimization and Lagrangian Multiplier Techniques

Simple Constrained Optimization

Web Chapter A discussed some of the techniques for solving unconstrained optimization problems. In this appendix the Lagrangian multiplier technique is developed to deal with some classes of constrained optimization problems. In Web Chapter B linear programming, a more general technique for dealing with constrained optimization, is developed.

Most organizations have constraints on their decision variables. The most obvious constraints, and the easiest to quantify and incorporate into the analysis, are the limitations imposed by the quantities of resources (such as capital, personnel, facilities, and raw materials) available to the organization. Other more subjective constraints include legal, environmental, and behavioral limitations on the decisions of the organization.

When the constraints take the form of equality relationships, classical optimization procedures can be used to solve the problem. One method, which can be employed when the objective function is subject to only *one* constraint equation of a relatively simple form, is to solve the constraint equation for one of the decision variables and then substitute this expression into the objective function. This procedure converts the original problem into an unconstrained optimization problem, which can be solved using the calculus procedures developed in this Web Chapter.

Example

Constrained Profit Maximization: Indiana Petroleum Company

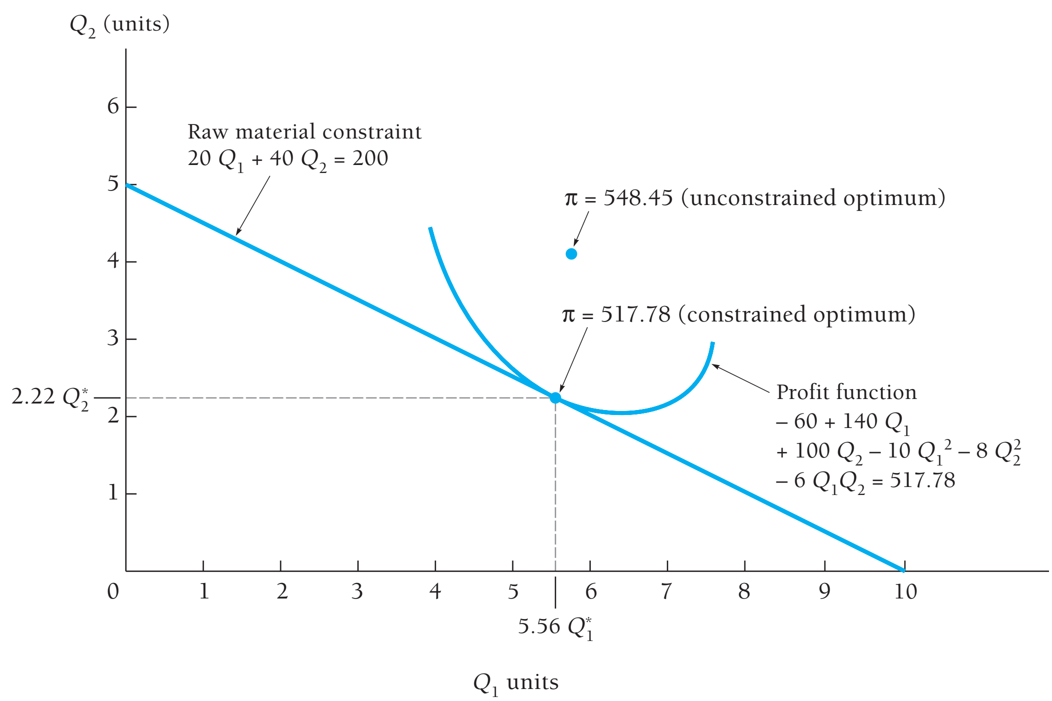
Consider again the two-product profit-maximization problem (Equation A.24) from Web Chapter A. Suppose that the raw material (crude oil) needed to make the products is in short supply and that the firm has a contract with a supplier calling for the delivery of 200 units of the given raw material during the forthcoming period. No other sources of the raw material are available. Also, assume that *all the raw material must be used during the period* and that none can be carried over to the next period in inventory. Furthermore, suppose that Product 1 requires 20 units of the raw material to produce one unit of output and Product 2 requires 40 units of raw material to produce one unit of output. The constrained optimization problem can be written as follows:

$$\text{Maximize } \pi = -60 + 140Q_1 + 100Q_2 - 10Q_1^2 - 8Q_2^2 - 6Q_1Q_2 \qquad \text{[AA.1]}$$

$$\text{subject to } 20Q_1 + 40Q_2 = 200 \qquad \text{[AA.2]}$$

The raw material constraint line and the profit function (curve) that is tangent to the constraint line are shown together in Figure AA.1. Note that the solution to the unconstrained problem obtained earlier—$Q_1 = 5.77$ and $Q_2 = 4.08$—is not a feasible solution to the constrained problem because it requires $20(5.77) + 40(4.08) = 278.6$ units of raw material when in fact only 200 units are available. Following the procedure just described, we solve the constraint for Q_1:

FIGURE AA.1 Constrained Profit Maximization: Indiana Petroleum Company



$$Q_1 = \frac{200}{20} - \frac{40Q_2}{20}$$
$$= 10 - 2Q_2$$

Substituting this expression for Q_1 in the objective function, we obtain

$$\begin{aligned}\pi &= -60 + 140(10 - 2Q_2) + 100Q_2 - 10(10 - 2Q_2)^2 \\ &\quad -8Q_2^2 - 6(10 - 2Q_2)Q_2 \\ &= -60 + 1400 - 280Q_2 + 100Q_2 - 1000 + 400Q_2 \\ &\quad -40Q_2^2 - 8Q_2^2 - 60Q_2 + 12Q_2^2 \\ &= 340 + 160Q_2 - 36Q_2^2\end{aligned}$$

Taking the derivative of this expression with respect to Q_2 yields

$$\frac{d\pi}{dQ_2} = 160 - 72Q_2$$

Setting $d\pi/dQ_2$ equal to zero and solving for Q_2, we obtain

$$0 = 160 - 72Q_2$$
$$Q_2^* = \frac{160}{72}$$
$$= 2.22 \text{ units}$$

In turn solving for Q_1, we obtain

$$Q_1^* = 10 - 2(2.22)$$
$$= 5.56 \text{ units}$$

Thus $Q_1^* = 5.56$ and $Q_2^* = 2.22$ is the optimal solution to the constrained profit-maximization problem.

Using the constraint to substitute for one of the variables in the objective function, as in the preceding example, will yield an optimal solution only when there is one constraint equation and it is possible to solve this equation for one of the decision variables. With more than one constraint equation and/or a complex constraint relationship, the more powerful method of Lagrangian multipliers can be employed to solve the constrained optimization problem.

Lagrangian Multiplier Techniques

The Lagrangian multiplier technique creates an additional artificial variable for each constraint. Using these artificial variables, the constraints are incorporated into the objective function in such a way as to leave the value of the function unchanged. This new function, called the Lagrangian function, constitutes an unconstrained optimization problem. The next step is to set the partial derivatives of the Lagrangian function for each of the variables equal to zero and solve the resulting set of simultaneous equations for the optimal values of the variables.

Example

Lagrangian Multipliers: Indiana Petroleum Company (continued)

The Lagrangian multiplier method can be illustrated using the example discussed in the previous section. First, the constraint equation, which is a function δ of the two variables Q_1 and Q_2, is rearranged to form an expression equal to zero:

$$\delta(Q_1, Q_2) = 20Q_1 + 40Q_2 - 200 = 0$$

Next we define an artificial variable λ (lambda) and form the Lagrangian function.[12]

$$L_\pi = \pi(Q_1, Q_2) - \lambda\delta(Q_1, Q_2)$$
$$= -60 + 140Q_1 + 100Q_2 - 10Q_1^2 - 8Q_2^2 - 6Q_1Q_2$$
$$-\lambda(20Q_1 + 40Q_2 - 200)$$

As long as $\delta(Q_1,Q_2)$ is maintained equal to zero, the Lagrangian function L_π will not differ in value from the profit function π. Maximizing L_π will also maximize π. L_π is seen to be a function of Q_1, Q_2, and λ. Therefore, to maximize L_π (and also π), we need to partially differentiate L_π with respect to each of the variables, set the partial deriva-

[12] To assist in the interpretation of the results, it is often useful to adopt the arbitrary convention that in the case of a *maximization* problem the lambda term should be *subtracted* in the Lagrangian function. In the case of a *minimization* problem, the lambda term should be *added* in the Lagrangian function.

tives equal to zero, and solve the resulting set of equations for the optimal values of Q_1, Q_2, and λ. The partial derivatives are equal to

$$\frac{\partial L_\pi}{\partial Q_1} = 140 - 20Q_1 - 6Q_2 - 20\lambda$$
$$\frac{\partial L_\pi}{\partial Q_2} = 100 - 16Q_2 - 6Q_1 - 40\lambda$$
$$\frac{\partial L_\pi}{\partial \lambda} = 20Q_1 - 40Q_2 + 200$$

Setting the partial derivatives equal to zero yields the equations

$$20Q_1 + 6Q_2 + 20\lambda = 140$$
$$6Q_1 + 16Q_2 + 40\lambda = 100$$
$$20Q_1 + 40Q_2 = 200$$

After solving this set of simultaneous equations, we obtain $Q_1^* = 5.56$, $Q_2^* = 2.22$, and $\lambda^* = +.774$. (*Note:* These are the same values of Q_1 and Q_2 that were obtained earlier in this section by the substitution method.)

If a problem has two or more constraints, then a separate λ variable is defined for each constraint and incorporated into the Lagrangian function. In general, λ measures the marginal change in the value of the objective function resulting from a one-unit change in the value on the righthand side of the equality sign in the constraint relationship. In the example above, λ^* equals \$.774 and indicates that profits could be increased by this amount if one more unit of raw material was available; that is, an increase from 200 units to 201 units. The λ values are analogous to the dual variables of linear programming, which are discussed in Web Chapter B.

EXERCISES

1. What purpose do the artificial variables (λs) serve in the solution of a constrained optimization problem by Lagrangian multiplier techniques?
2. What do the artificial variables (λs) measure in the solution of a constrained optimization problem using Lagrangian multiplier techniques?
3. Bounds, Inc. has determined through regression analysis that its sales (S) are a function of the amount of advertising (measured in units) in two different media. This is given by the following relationship (X = newspapers, Y = magazines):

$$S(X,Y) = 200X + 100Y - 10X^2 - 20Y^2 + 20XY$$

Assume the advertising budget is restricted to 20 units.

a. Determine (using Lagrangian multiplier techniques) the level of newspaper and magazine advertising that maximizes sales subject to this budget constraint.
b. Calculate the firm's sales at this constrained optimum level.
c. Give an economic interpretation for the value of the Lagrangian multiplier (λ) obtained in part (a).
d. Compare the answer obtained in parts (a) and (b) above with the optimal solution to the *unconstrained* problem in Exercise 13 of Web Chapter A.